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RESOLVIENDO ECUACIONES CÚBICAS CON TRIÁNGULOS: UN ENFOQUE GEOMÉTRICO PARA LA SELECCIÓN DE RAÍCES

Solving Cubic Equations with Triangles: A Geometric Approach to Root Selection

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Resumen— En muchas situaciones prácticas, los ingenieros se enfrentan a problemas que conducen a ecuaciones cuadráticas. Estas tienen dos posibles soluciones, y generalmente es sencillo determinar cuál de ellas es relevante, utilizando principios físicos. Sin embargo, algunos problemas conducen a ecuaciones cúbicas, que son más complejas. Mientras que algunas cúbicas tienen una raíz real y dos raíces complejas conjugadas, otras presentan tres soluciones reales distintas. Esto plantea una pregunta natural e importante: ¿cómo podemos decidir cuál de las raíces corresponde a la realidad física del problema? En este artículo se presenta una herramienta geométrica —el método del triángulo equilátero— para ayudar a identificar la solución físicamente significativa de una ecuación cúbica. Se ilustra este método utilizando tres problemas de ingeniería que surgieron en una investigación propia a saber: (1) la obtención de una ecuación cinética para la remoción de magnesio en aluminio líquido, (2) el análisis del comportamiento de un material magnetocalórico, y (3) la determinación de la eficiencia en una pala de un aerogenerador.

Palabras clave— Aerogeneradores, Cinética química, Ecuaciones cúbicas, Materiales magnetocalóricos, Modelado matemático.

Abstract— In many practical situations, engineers encounter problems that lead to quadratic equations. These equations have two possible solutions, and it is usually straightforward to determine which one is relevant using physical principles. However, some problems lead to cubic equations, which are more complex. While some cubic equations have one real root and two complex conjugate roots, others possess three distinct real solutions. This raises a natural and important question: how can we decide which of these roots corresponds to the physical reality of the problem?. In this article, a geometric tool—the equilateral triangle method—is presented to help identify the physically meaningful solution of a cubic equation. This method is illustrated through three engineering problems that arose during the course of the author's research: (1) the derivation of a kinetic equation for the removal of magnesium from liquid aluminum, (2) the analysis of the behavior of a magnetocaloric material, and (3) the determination of efficiency in a wind turbine blade.

Keywords— Wind turbines, Chemical kinetics, Cubic equations, Magnetocaloric materials, Mathematical modeling.

I. INTRODUCCION

In engineering problems, it is common to encounter mathematical models that lead to quadratic equations. These typically yield two real solutions, and in most cases, physical principles allow for the straightforward identification of the solution that corresponds to the real-world behavior of the system. However, in certain cases, especially in more complex engineering scenarios, the governing equations take the form of cubic equations. These can yield up to three distinct real roots, making it more challenging to determine which solution accurately represents the physical reality of the problem. This situation requires the application of an additional method to identify the root that aligns with the actual physical conditions. In this work, a geometric method is presented to discern which of the real solutions of a cubic equation corresponds to the physical behavior of the system. The approach is based on a geometric interpretation of the trigonometric representation of the roots of a cubic equation. The method is illustrated through three engineering cases that I have encountered during the course of my research.

II. RESOLUTION OF ALGEBRAIC CUBIC EQUATION

Before presenting the three cases, I will first explain the algebraic method for solving a general cubic equation.

$$ax^3 + bx^2 + cx + d = 0 \tag{1}$$

where $a \neq 0$. It is important to clarify that the coefficients of the equation are real numbers. This equation may have either one real solution and two complex conjugate solutions, or three real solutions. Before presenting the solution, it is important to note that we can express the roots in the form.

$$\begin{aligned} x_1 &= f_1(a,b,c,d) \\ x_2 &= f_2(a,b,c,d) \\ x_3 &= f_3(a,b,c,d) \end{aligned} \tag{2}$$

Where each x_i is a solution of the cubic equation. However, although we present them this way, each of these expressions can be transformed into the others. This means that, depending on how we arrange the expressions, the set of solutions could appear as: x_1, x_3, x_2 or x_2, x_3, x_1 or x_3, x_1, x_2 and so on. These permutations can be interpreted geometrically in terms of the symmetries of an equilateral triangle, a viewpoint that connects with classical geometric interpretations of cubic equations [1]. The geometric interpretation used in this work was developed independently. For example, the first permutation mentioned corresponds to a mirror symmetry, while the others correspond to rotations of 120 degrees either clockwise or counterclockwise. To solve this type of equation, the equilateral triangle method can be used, which is shown in the figure 1.

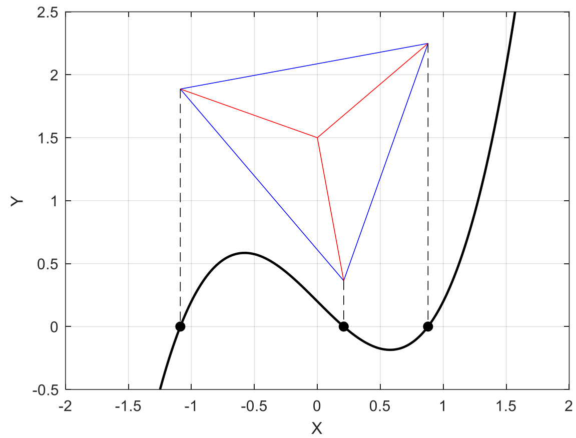

Fig. 1. Equilateral triangle construction used to obtain the solution to the cubic.

In Fig. 1, an equilateral triangle is shown, where the projections of its vertices, indicated by the dashed vertical lines, onto the x-axis correspond to the roots of the cubic equation. The black curve represents a cubic equation, and the black points correspond to the real roots of the polynomial, that is, the points where the curve intersects the x-axis. Similar geometric interpretations of the roots of cubic equations have been discussed in the mathematical literature [1]. The possible permutations of the roots are associated with the symmetry operations of the triangle (rotations and reflections). Therefore, the solutions can be expressed as:

$$\begin{aligned} x_1 &= cen + R\cos\theta \\ x_2 &= cen + R\cos\left(\theta + \frac{2\pi}{3}\right) \\ x_3 &= cen + R\cos\left(\theta - \frac{2\pi}{3}\right) \end{aligned} \tag{3}$$

where cen represents the horizontal shift of the triangle center, R is the radius of the circumscribed circle, and θ determines the orientation of the triangle. It can be observed that this set of solutions satisfies all the symmetries expected from the cubic equation's solution formula. The next step is to determine the values of each of the parameters. To do this, we note that cen is given by the expression:

$$cen = -\frac{b}{3a} \tag{4}$$

Using the substitution

$$x = y - \frac{b}{3a} \tag{5}$$

and substituting Eq. (5) into Eq. (1), the cubic equation can be transformed into the reduced form

$$y^3 + py + q = 0 \tag{6}$$

where p and $q$are given by

$$p = \frac{3ac - b^2}{3a^2}, \qquad (7)$$
$$q = \frac{2b^3 - 9abc + 27a^2 d}{27a^3}$$

From Eq. (6), the relationships between the roots and the coefficients of the polynomial can be written as:

$$y_1 y_2 + y_1 y_3 + y_3 y_2 = p \qquad (8)$$
$$-y_1 y_2 y_3 = q$$

We have that the cosine function appearing in the Eq. (3) can be transformed using Euler's identity.

$$cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \qquad (9)$$

Using this representation, the solutions of the cubic equation can also be written in the form:

$$y_1 = u_1 + u_2 \qquad (10)$$
$$y_2 = \alpha_1 u_1 + \alpha_2 u_2$$
$$y_2 = \alpha_2 u_1 + \alpha_1 u_2$$

Where:

$$u_1 = \frac{Re^{i\theta}}{2} \qquad (11)$$
$$u_2 = \frac{Re^{-i\theta}}{2}$$

and

$$\alpha_1 = e^{i\frac{2\pi}{3}} \qquad (12)$$
$$\alpha_2 = e^{-i\frac{2\pi}{3}}$$

Substituting the expressions given in Eqs. (10) and (12) into the relationships between the roots and the coefficients of the cubic equation, Eq. (8), we obtain

$$-3u_1 u_2 = p \qquad (13)$$
$${u_1}^3 + {u_2}^3 = -q$$

Combining these equations gives us that, to obtain the terms u_1 and u_2, the following equation is obtained.

$$u_i^6 + q u_i^3 - \frac{p^3}{27} = 0 \qquad (14)$$

Which can be solved by means of the well-known general formula for the quadratic equation. Of the term inside the square root, we know that the cubic equation has one real solution and two complex solutions or has three real solutions.

Once the quantities $u_1$and $u_2$are determined as shown in eq. 11, the eq. (13) can be rewritten in terms of $R$and θ.

$$R^2 = -\frac{4p}{3} \qquad (15)$$
$$\frac{R^3}{4}\cos(3\theta) = -q$$

We note that the function arccos (the inverse of the cosine) yields values in the interval [0,π]. As an engineer, In applied research related to mathematical modeling of physical processes, it is common to encounter cubic equations whose solutions must be carefully analyzed. In particular, situations often arise where the cubic equation has three real solutions, posing the challenge of identifying which root corresponds to the physically meaningful behavior of the system. Although mathematically all three solutions are valid, only one represents the actual physical state. The key to selecting the correct solution lies in ensuring continuity with respect to the physical parameters involved. This concept will be illustrated through three engineering examples drawn from practical applications.

III. ENGINEERING CASE STUDIES

A. First Example: Magnesium Removal from Aluminum Alloys Using Reactive Powder Injection of Na_2SiF_6

The first example corresponds to the case of magnesium removal in aluminum alloys by injecting reactive powders of sodium hexafluorosilicate Na_2SiF_6. This study was carried out by E. Macias and A. Flores [2][3]. The importance of magnesium removal from aluminum alloys lies in the fact that aluminum cans constitute the most readily available source of aluminum scrap. However, this scrap material is composed of Al-Mg alloys containing magnesium content of up to 3–5% by weight. When this scrap is used for producing casting aluminum alloys such as A319 or A380, which require a maximum magnesium content of 0.1%, it becomes necessary to remove the excess magnesium. In the process of magnesium removal from liquid aluminum, sodium silicofluoride was used as the reactive agent. The kinetic analysis of this process leads to a mathematical model in which the governing equation takes the form of a cubic equation. Solving this equation allows the evolution of the unreacted core radius inside the reacting particle to be determined as a function of time, which is essential for predicting the efficiency of the magnesium removal process. The figure 2 shows the injection of the powder and the locations where the reaction takes place.

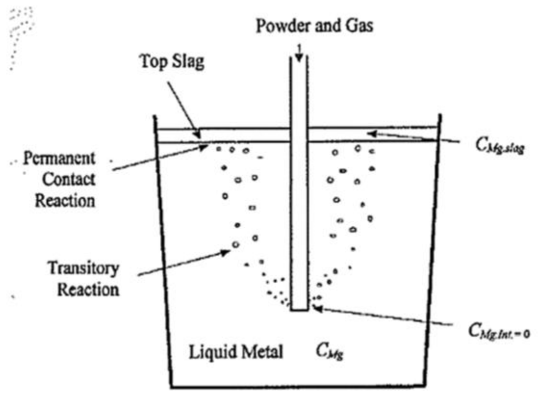

Fig. 2. Schematic representation of the reactions sites established during submerged powder injection [3]

Inside the reactor, there are several reaction sites:

• The reaction between magnesium and the free particles discharged through the lance,
• The reaction with particles trapped in gas bubbles, the reaction with the released SiF_4 gas,
• And the reaction with the slag floating on the metal bath.

For this study, only the reaction involving free particles in the liquid aluminum will be analyzed. The Fig. 3 illustrates the reaction model through a single particle [2]-[5], showing the radius of the unreacted core and the radius of the particle itself. In this model, the reactants reach the interface between the product layer and the unreacted core. The Fig. 3 shows the successive stages in the kinetic model:

• Transport of reactants from the bulk liquid to the particle through the boundary layer.
• Transport of reactants through the product layer to the reaction interface.
• Chemical reaction at the interface between the product layer and the unreacted core.

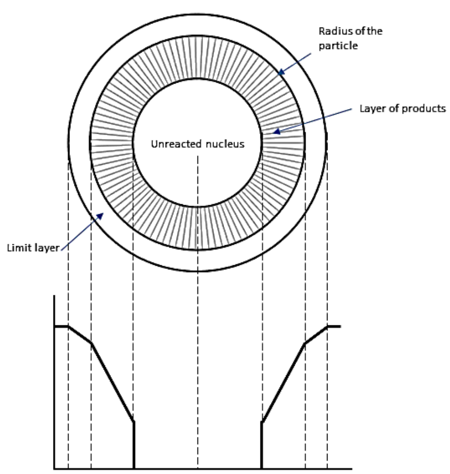

Fig. 3. Representation of reactant concentration in a solid particle that does not change in size during the reaction, where a product layer forms around its outer surface [4].

Table 1.
Nomenclature used in the first example.

Nomenclature	Meaning
A_P	Reaction area = $4\pi r_P^2$, m^2
Bi_m	Biot number $=k\, d_P / D_{ef}$
C	Volumetric concentration, mol/m^3
C_i	volumetric concentration of Mg at limit layer, mol/m^3
D_{ef}	effective diffusion, m^2/s
J	Molar flux rate, mol/s
d_P	particle diameter, m
k	Mass transfer coefficient, m/s
r	radial coordinate, m
r_c	radius of the unreacted nucleus, m
r_P	particle's radius, m
α	stoichiometric coefficient
ρ_s	molar density of the product layer, mol/m^3
τ	total particle conversion time, s

To describe the kinetic model quantitatively, several variables and parameters are introduced. For clarity, the symbols used in the formulation of the model are summarized in Table 1, where their physical meaning and units are specified.

From this formulation, a kinetic equation describing the evolution of the radius of the unreacted core is derived, which ultimately leads to a cubic equation for the dimensionless core radius.

Due to the high process temperature, it is considered that the kinetics are controlled by mass transfer trough limit layer and product layer. We now present the chemical kinetics equations. The transfer of reactants through the boundary layer is given by:

$$J = kA_P(C - C_i) \tag{16}$$

The transfer of reactants through the product layer, which can be determined by solving Fick's First Law under quasi-steady-state conditions. Fick's First Law can be expressed as:

$$J = 4\pi r^2 D_{ef} \frac{\partial C}{\partial r} \tag{17}$$

The conditions at the frontier are:

$$\begin{aligned} r &= r_P \qquad C = C_i \\ r &= r_c \qquad C = 0 \end{aligned} \tag{18}$$

If the equation corresponding to Fick's law (Eq.17) is integrated. We have that:

$$C_i = \frac{J}{4\pi D_{ef}}\left[\frac{1}{r_c} - \frac{1}{r_P}\right] \tag{19}$$

By combining Eqs. (17) and (19), which describe magnesium transport through the boundary layer and through the product layer, respectively, and eliminating the interfacial concentration C_i, the following expression is obtained.

$$C = \frac{J}{4\pi D_{ef}}\left[\frac{1}{r_c} - \frac{1}{r_P}\right] + \frac{J}{4\pi r_P^2 k} \quad (20)$$

The flux of magnesium at the reaction interface is directly related to the rate of shrinkage of the unreacted core. This can be related to the growth of the product layer in the particle through the following equation.

$$J = -\alpha \rho_s 4\pi r_c^2 \frac{dr_c}{dt} \quad (21)$$

From this, the growth rate of the reaction product layer can be obtained. The equation is integrated between the limits t=0 and t_r, where t_r is the residence time.

$$t = \frac{\alpha \rho_s r_P^2}{6 D_{ef} C}\left[3\left(1 - \frac{r_c^2}{r_P^2}\right) - 2\left(1 - \frac{r_c^3}{r_P^3}\right)\left(1 - \frac{D_{ef}}{r_P k}\right)\right] \quad (22)$$

The total reaction time corresponds to the condition $r_c = 0$, which represents complete conversion of the particle. Under this condition the total reaction time is given by

$$\tau = \frac{\alpha \rho_s d_P^2}{24 D_{ef} C}\left[1 - \frac{4}{Bi_m}\right] \quad (23)$$

Introducing the dimensionless variable

$$x = \frac{r_c}{r_P} \quad (24)$$

and rearranging the previous expressions, the governing equation can be written as the cubic equation

$$2\frac{Bi_m - 2}{Bi_m + 4}x^3 - 3\frac{Bi_m}{Bi_m + 4}x^2 + \left(1 - \frac{t}{\tau}\right) = 0 \quad (25)$$

The cubic equation may admit three real solutions. In such cases, it is necessary to determine which of these solutions corresponds to the physically meaningful one. To analyze this situation, let us consider the particular case $t = \tau$ (Eq. 25), which represents the condition where the reaction has reached completion. If we represent it using the equilateral triangle method, we would have something like this (considering that $Bi_m > 2$). Setting $t = \tau$ in Eq. (25), which corresponds to the condition of complete reaction, the cubic equation can be factorized to identify its roots. The equation can then be written as:

$$\left[2\frac{Bi_m - 2}{Bi_m + 4}\right]x^2\left(x - \frac{3}{2}\frac{Bi_m}{Bi_m - 2}\right) = 0 \quad (26)$$

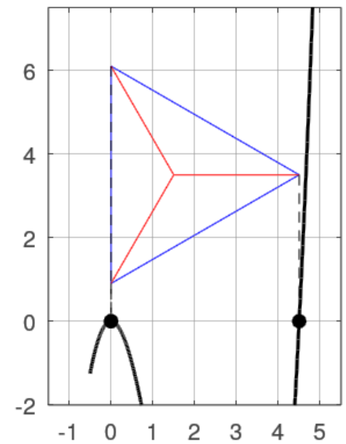

Fig. 4. Equilateral triangle construction used to obtain the solution to the cubic for the first example considering that time is equal to total reaction total. The solid black curve represents the cubic equation (26), while the black points indicate its real roots.

In this configuration two of the roots of the cubic equation are located at $x = 0$. Using the trigonometric representation of the roots given in Eq. (3), the geometric parameters of the triangle can be determined directly. With this orientation, we ensure that two of the roots of the cubic equation are zero, and we also find the center and the radius R of the circle circumscribed about the equilateral triangle.

$$R = \frac{Bi_m}{Bi_m - 2} \quad (27)$$
$$cen = \frac{Bi_m}{2(Bi_m - 2)}$$

At the total reaction time $t = \tau$, the reaction is complete and the unreacted core radius becomes zero ($r_c = 0$), which implies $x = 0$. For this reason, two of the roots of the cubic equation coincide at $x = 0$, while the third root does not correspond to the physical state of the system. To distinguish which of the two possible solutions it can be, we simply consider the case when $0 < t < \tau$ and we have something like this, where $\theta > 0$. From here, we have that the solution to the cubic equation must be (since it would be the only one of the two possible solutions that yields a positive value).

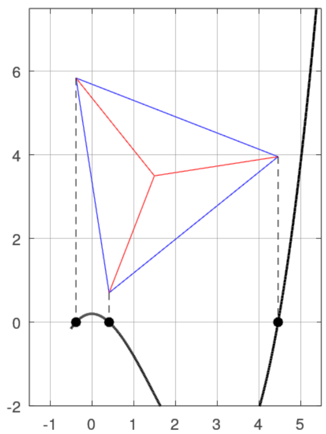

Fig. 5. Equilateral triangle construction used to obtain the solution to the cubic for the first example considering that time is less than the total reaction total. The solid black curve represents the cubic equation (25), while the black points indicate its real roots.

To apply the trigonometric representation of the roots, the cubic equation (25) is first transformed into the reduced form given in Eq. (6). From this transformation, the solution can be expressed in terms of the angle θ.

$$x = \frac{Bi_m}{Bi_m - 2}\left[\frac{1}{2} + \cos\left(\theta - \frac{2\pi}{3}\right)\right] \tag{28}$$

The value of θ is obtained by applying the transformation in the equation. This approach can be made when Bim < 2, but it would lead to the same solution. After calculating the value of q for this equation, the angle θ can be determined from:

$$cos3\theta = 1 - 2\frac{(Bi_m + 4)(Bi_m - 2)^2}{Bi_m^3}\left(1 - \frac{t}{\tau}\right) \tag{29}$$

It should be noted that the angle θ obtained from Eq. (29) corresponds to the root that was discarded in the previous analysis, namely the one that does not become zero when $t = \tau$.

B. Second Example: Magnetocaloric Refrigeration

Cooling technology in modern society has become essential, with applications ranging from the food industry and healthcare sector to the energy sector. The predominant cooling technology is based on vapor compression, which is characterized by low energy efficiency and the use of refrigerant gases that are harmful to the environment. Therefore, alternative technologies are required that are both energy-efficient and environmentally friendly. One such alternative is cooling based on the magnetocaloric effect. Magnetocaloric cooling technology relies on materials that undergo a sharp transition to the ferromagnetic state, during which a sudden release of heat occurs. This phenomenon happens when the material is subjected to an external magnetic field, causing the magnetic moments within the material to align [6]-[7]. Regarding magnetic refrigeration, I have been collaborating at CIDESI for the LANITEF project since 2022. To understand the operation of this equipment, it is necessary to know how the magnetocaloric material responds to the application of a magnetic field and to determine its degree of magnetization. The thermodynamic analysis of this behavior leads to a cubic equation that relates the magnetization of the material to the temperature and the applied magnetic field. Solving this equation is essential for predicting the magnetic response of the material and for evaluating the performance of magnetocaloric refrigeration systems.

To formulate the mathematical model describing the magnetocaloric behavior of the material, several physical variables and parameters are required. The symbols used in this second example, together with their physical meaning, are summarized in Table 2.

Table 2.
Nomenclature used in the second example.

Nomenclature	Meaning
B	magnetic induction field, T
J	total angular momentum quantum number =3.5
M	Magnetic moment, J/mol T
M_j	Atomic magnetic moment =$g_L J \mu_B$,J/T
T	Temperature, K
T_C	Curie's Temperature =293 K
k_B	Boltzmann constant
g_L	Lande's factor =2
w	Molecular field coefficient
μ_B	Bohr magneton = 9.3 e-24 J/T

According to the literature [7], it is known that:

$$M = NM_jB_j(x) \tag{30}$$

Where $B_j(x)$ is the function of Brillouin

$$B_j(x) = \frac{2J+1}{2J}coth\left(\frac{2J+1}{2J}x\right) - \frac{1}{2J}coth\left(\frac{x}{2J}\right) \tag{31}$$

To define the behavior of a ferromagnetic material, such as gadolinium, I followed a procedure in which the magnetization field is incorporated explicitly into the function used for the calculation. The solution corresponds to the intersection between the anhysteretic magnetization curve and a straight line representing the effective field.

$$x = \frac{M_j(B + wM)}{k_BT} \tag{32}$$

Figure 6 shows the point where the two curves intersect to calculate the magnetic moment, for B = 2 T and a temperature of 200 K. In the vicinity of x = 0, the following cubic equation can be formulated and solved (for this case, it was assumed that the cubic equation can be solved when x < 1.32; if x were greater, the system of equations formed by equations would need to be solved numerically).

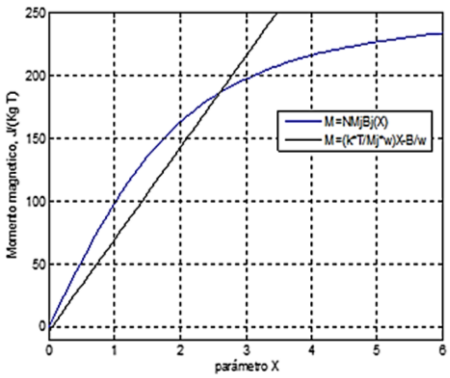

Fig. 6. Line intersection method to calculate the magnetic moment using Equation (30) and (32) for B = 2 T and T = 200 K [6]

For certain parameter values, the system yields a unique solution. However, for other values, three real solutions may exist. In these cases, the physically meaningful solution is selected by enforcing continuity along the magnetization curve as the external field varies. This path continues until a critical point is reached, where the system abruptly transitions to a different branch of solutions, resulting in a discontinuous jump. As the external field is further varied, the system can return to the original branch, effectively producing a hysteresis loop.

We proceed to expand the Brillouin function into a Taylor series, keeping terms up to the third order.

$$a\coth(ax) = \frac{1}{x} - \frac{a^2}{3}x - \frac{a^4}{45}x^3 + \ldots \tag{33}$$

Therefore, the Brillouin function can be expressed as:

$$B_j(x) = \frac{J+1}{3J}x + \frac{[J+1][(J+1)^2+J^2]}{90J^3}x^3 + \ldots \tag{34}$$

To obtain an analytical expression for the magnetization, Eqs. (30) and (32) are equated. The Eq. (30) contains the Brillouin function, the latter is expanded in a Taylor series.

$$\frac{k_B T}{w M_j}x - \frac{B}{w} = N M_j\left[\frac{J+1}{3J}x - \frac{[J+1][(J+1)^2+J^2]}{90J^3}x^3\right] + \ldots \tag{35}$$

Retaining terms up to third order yields the following expression:

$$\frac{(J+1)^2+J^2}{30J^2}T_C x^3 + (T - T_C)x - \frac{M_j B}{k_B} = 0 \tag{36}$$

Where T_C is the Curie temperature (which indicates the transition from ferromagnetic to paramagnetic behavior of the material) and is given by the expression:

$$T_C = \frac{N M_j^2 (J+1)}{3 k_B J} w \tag{37}$$

If we examine the cubic equation, we find that three real solutions can occur only when $T < T_C$. As illustrated in Fig. 6, the Curie temperature corresponds to the condition where the slope of the straight line representing Eq. (32) becomes equal to the slope of the curve given by Eq. (30) at $x = 0$. At this point the two curves are tangent, and the system transitions from a single solution to the possibility of multiple solutions. This same condition is reflected in the cubic equation obtained from the Taylor expansion. This condition is also consistent with the classical discriminant criterion for cubic equations [8], and it can also be derived directly from the expressions given in Eqs. (10) and (14). For a reduced cubic equation $x^3 + px + q = 0$, the discriminant $\Delta = -4p^3 - 27q^2$ determines the number of real solutions, with three real roots occurring when $\Delta > 0$. When $T > T_C$, the cubic equation admits only one real solution. In the reduced form of the cubic, it can be easily verified that the discriminant is always negative.

If we examine the cubic equation, we find that only in the case when $T < T_C$ (The cubic equation can have three real solutions). Once again, it becomes necessary to determine which of these three corresponds to the physical solution. To analyze this, let's first consider the case when B=0, meaning no magnetic field is applied to the material. In this situation, the roots of the cubic equation consist of one root equal to zero, one positive root, and one negative root. In this situation, the triangle representation simplifies to:

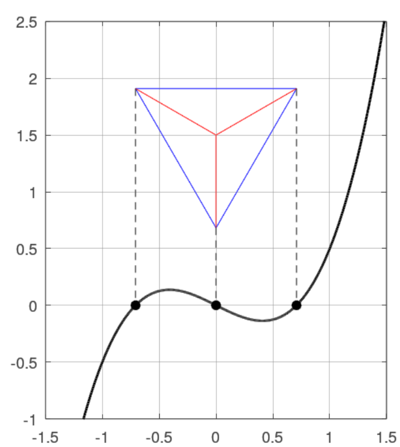

Fig. 7. Equilateral triangle construction used to obtain the solution to the cubic for the second example considering that the magnetic camp is zero. The solid black curve represents the cubic equation (36) which has been written in the

reduced form given in Eq. (6), while the black points indicate its real roots.

In this case, we will consider that the magnetocaloric material is magnetized and aligned with the direction of the applied magnetic field. We take as the solution one of the values that correspond to the magnetized state of the magnetocaloric material. Looking at the solution when $B = 0$, Eq. (36) implies that $q = 0$. Substituting this result into Eq. (15) leads to:

$$cos(3\theta) = 0 \quad \Rightarrow \quad \theta = 30^\circ \tag{38}$$

Then, the solution is given by:

$$x = R\cos\theta \tag{39}$$

And this solution holds for values of B greater than zero. An interesting case to consider is what happens when the magnetocaloric material is exposed to progressively decreasing magnetic fields, all the way down to the point where the direction of the magnetic field is reversed. Here, there would be two positive solutions and one negative solution, but only one of them ensures continuity. As the magnetic field intensity increases in the direction opposite to the magnetization, the magnetization eventually jumps to a negative value. As the magnetic field intensity increases in the direction opposite to the magnetization, the magnetization eventually jumps to a negative value. This magnetic hysteresis phenomenon is more complex, but this represents a first approximation [9]. Then, the solution is given by:

$$x = R\cos\left(\theta + \frac{2\pi}{3}\right) \tag{40}$$

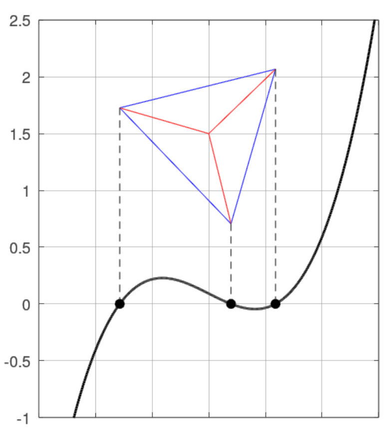

Fig. 8. Equilateral triangle construction used to obtain the solution to the cubic for the second example considering that the magnetic camp is different to zero.

C. Third example. Wind Turbine Blade

The design of wind turbine blades requires the prediction of aerodynamic performance under different operating conditions. Mathematical models are therefore used to estimate how the wind interacts with the rotor and how much energy can be extracted from the airflow. These models allow engineers to determine key parameters that influence the efficiency of the turbine.

The axial induction factor a plays a fundamental role in the aerodynamic analysis of wind turbine blades, since it quantifies the reduction of the wind velocity as the air passes through the rotor disk due to energy extraction. Determining the value of this factor is essential for estimating the power that can be extracted from the wind and for predicting the aerodynamic performance of the turbine. By combining the relations that describe the axial and tangential velocity components with the momentum balance of the flow, a mathematical expression for the induction factor can be obtained. This formulation ultimately leads to a cubic equation in the variable a, whose physically meaningful solution must be identified.

To describe the aerodynamic model of the wind turbine blade, several variables and parameters are introduced. For clarity, the symbols used in the formulation of this third example, together with their physical meaning and units, are summarized in Table 3.

Table 3.
Nomenclature used in the third example.

Nomenclature	Meaning
P	Total power, W
M	Magnetic moment, J/mol T
V_0	Wind speed, m/s
a	axial induction factor
a'	tangential induction factor
r	Radial coordinate, m
u_z	velocity z at blades plane, m/s
u_θ	Velocity θ at blades plane, m/s
x	$\omega r/V_0$
ρ	Air density, kg/m^3
ω	Rotation of blades, rad/s

For wind turbine blade design purposes, several simplifying assumptions are made. First, the region where the blades operate is modeled as a porous zone [10]-[11]. As energy is extracted from the wind, this zone slows down the airflow, leading to a reduction in wind velocity across the rotor area. An axial induction factor a is used to define the velocity at blades plane:

$$u_z = (1 - a)V_0 \tag{41}$$

Similarly, in this region, a wake is generated that includes a rotational velocity component in the downstream flow using tangential induction factor a'.

$$u_\theta = 2a'\omega r \tag{42}$$

All these factors are used to calculate the power extracted from the wind by the wind turbine. For a single blade, the power would be given by:

$$dP = (2\pi r \rho u_z)\, r\omega u_\theta\, dr \qquad (43)$$

The total power is found by integrating dP from 0 to R.

$$P = 4\pi\rho\omega^2 V_0 \int_0^R a'(1-a)\, r^3\, dr \qquad (44)$$

The goal is to maximize the power generated by the wind turbine, which requires maximizing the term inside the integral.

$$f(a', a) = a'(1-a) \qquad (45)$$

We obtain that:

$$\frac{df}{da} = (1-a)\frac{da'}{da} - a' = 0 \qquad (46)$$

If the local angles of attack are below stall, a and a' are not independent since the reacting force according to potential flow theory is perpendicular to the local velocity [10]. Reviewing the figure, we see that there are two similar triangles, which would satisfy the following equation:

$$\frac{a'\omega r}{aV_0} = \frac{V_0(1-a)}{\omega r(1+a')} \qquad (47)$$

From this, we can arrive at the following equation between a and a':

$$x^2 a'(1+a') = a(1-a) \qquad (48)$$

Where:

$$x = \frac{\omega r}{V_0} \qquad (49)$$

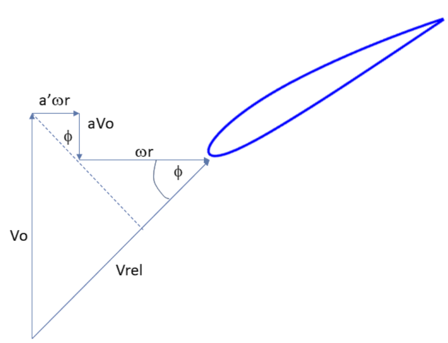

Fig. 9. Velocity triangle at the wind turbine blade showing the axial and tangential velocity components [10].

The previous equation can be differentiated, yielding the following:

$$x^2(1+2a')\frac{da'}{da} = 1 - 2a \qquad (50)$$

From here, by combining with the power optimization equation, we can arrive at the following relationship between a and a':

$$a' = \frac{1-3a}{4a-1} \qquad (51)$$

We arrive at the following equation:

$$16a^3 - 24a^2 + (9 - 3x^2)a + (x^2 - 1) = 0 \qquad (52)$$

If the following substitution is made

$$a = y + 1/2 \qquad (53)$$

The following equation is obtained

$$y^3 - 3/16(1+x^2)y - 1/32(1+x^2) = 0 \qquad (54)$$

By applying the triangle solution method for cubic equations, the parameters $R$and θ, associated with the geometric representation of the roots, can be obtained from Eq. (15) using the parameters of Eq. (54), which is already written in the reduced form given in Eq. (6).

$$R = \frac{1}{2}\sqrt{1+x^2} \qquad (55)$$

$$\cos(3\theta) = \frac{1}{\sqrt{1+x^2}} \qquad (56)$$

Since x^2 is always positive, the expression involving cos(3θ) will always have a value of θ that satisfies it, which means the cubic equation will always have three real solutions. To do this, it is necessary to determine which root has physical meaning. For this, let's consider the case when x tends to a very large value; here, we would have that:

$$cos(3\theta) \to 0 \quad \Rightarrow \quad \theta \to 30^\circ \quad \Rightarrow \quad R \to \infty \qquad (57)$$

Two of the solutions would tend toward very high values (because R approaches a very large value), which would not be physically realistic. Two of the solutions would tend toward very large values, since in this particular case the parameter Rapproaches infinity. For visualization purposes, however, the triangle shown in the figure is constructed using Eq. (3) with $cen = 0$ and a finite value of R. This allows the geometric representation of the roots to be displayed, even though mathematically Rtends to infinity in this limit.

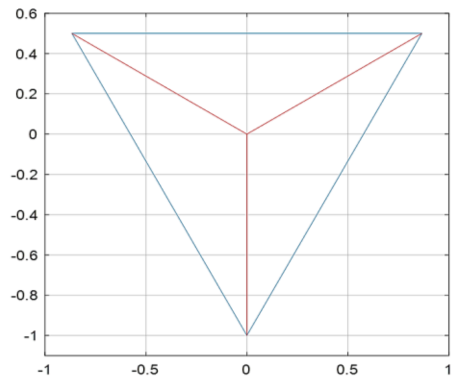

Fig. 10. Equilateral triangle construction used to obtain the solution to the cubic for the third example considering that case when *x* tends to a very large value.

To do this, it is necessary to determine which root has physical meaning. Let's consider the case when x tends to a very large value; here, we would have that only one of the three real roots remains bounded. This is important because it is the only solution that does not diverge to extremely large values, which would be unphysical in the context of the problem.

$$a = \frac{1}{2} + R\cos\left(\theta - \frac{2\pi}{3}\right) \tag{58}$$

To conclude the analysis, let us examine the case where x → ∞, From a physical standpoint, this scenario would not occur; however, it indicates the direction toward which the solution converges. Here, as 3θ approaches to 90, the cosine of this angle equals the sine of its complementary angle.

$$\sin\varphi = \cos 3\theta \tag{59}$$

The solution angle deviates from $\frac{\varphi}{3}$ from $-\frac{\pi}{2}$, therefore it follows that:

$$\cos\left(\theta - \frac{2\pi}{3}\right) = \cos\left(-\frac{\pi}{2} - \frac{\varphi}{3}\right) \tag{60}$$

For small values of φ, we have:

$$\sin\left(\frac{\varphi}{3}\right) \to \frac{\varphi}{3} \quad (\text{for small } \varphi) \tag{61}$$

We have that

$$\varphi \to \frac{1}{\sqrt{1+x^2}} \tag{62}$$

Therefore, it follows that

$$R\cos\left(\theta - \frac{2\pi}{3}\right) \to \left(\frac{1}{2}\sqrt{1+x^2}\right)\left(\frac{1}{3\sqrt{1+x^2}}\right) = \frac{1}{6} \tag{63}$$

And the result is

$$a \to \frac{1}{3} \tag{64}$$

As the rotational speed *ω* and thus also $x = \frac{\omega r}{V_0}$ is increased the optimum value for a tends to 1/3, which is consistent with the simple momentum theory for an ideal rotor.

IV. CONCLUSIONS

The equilateral triangle method introduced in this article provides a practical and visual approach to determine the physically meaningful root of a cubic equation.

Through the application of this method to three distinct engineering scenarios, it has been demonstrated that geometric insight can complement analytical solutions and reduce ambiguity in interpretation.

The proposed method is based on the ability to construct an equilateral triangle whose projection can indicate the roots of a cubic equation. The objective is to identify, among the three possible real roots, the one that represents an analytic continuation of the case in which the equation has only a single real root.

Typically, this physically meaningful solution can be determined by evaluating a limiting case. For example:

In the chemical reaction case, the solution was identified by analyzing the scenario of complete reaction.

In the magnetocaloric material problem, the limit was taken as the magnetic field approached zero.

In the wind turbine blade case, the limit was considered for a very large blade.

This approach often allows one of the three real roots to be discarded. Then, by examining situations slightly outside the limiting case—where the triangle may rotate or change in size (as in the wind turbine blade equation)—it becomes possible to discern which root corresponds to physical reality.

In this work, three engineering problems were analyzed in which cubic equations arise and a method was proposed to identify the physically meaningful solution. However, cubic equations appear in many other situations across science and engineering.

For example, they emerge when determining principal stresses in three-dimensional stress analysis, where the eigenvalues of the stress tensor are found by solving a cubic characteristic equation. In optics of anisotropic media, cubic equations arise when calculating the principal refractive indices of anisotropics materials. Similarly, in thermodynamics of real gases, the Van der Waals equation leads to a cubic equation in volume.

This geometric approach has the potential to serve as an educational tool for both students and professionals, promoting deeper understanding of mathematical models in physical systems. It should be noted that the geometric method is particularly useful in cases where the cubic equation admits three real solutions, allowing the physically meaningful root to be identified through geometric interpretation.

However, the selection of the correct root generally requires physical insight into the system being modeled. In the chemical kinetics example and in the wind turbine blade model, the physically meaningful solution was identified by examining appropriate limiting cases of the system. In the wind turbine case, additional analysis was required because the parameters of the cubic equation may take very large values, making the geometric representation less direct. In the magnetocaloric material example, the correct branch of the solution must be selected by considering the continuity of the magnetization curve, and when the magnetic field reverses direction the other roots must also be examined to properly describe the physical behavior of the system.

Although this situation was not treated in the present article, cubic equations also arise when determining the principal stresses in three-dimensional stress analysis, where the eigenvalues of the stress tensor are obtained by solving a cubic characteristic equation. In this case, the three real roots correspond to the three principal stresses, each of which has a clear physical meaning. This illustrates that the purpose of the proposed method is not to eliminate roots arbitrarily, but rather to help interpret which root corresponds to the physical phenomenon being modeled in each particular case.

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